

# A Note Concerning von Neumann Projector Measurement Pointer States, Weak Values, and Interference

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## Abstract

It is well known that the pointer state  $|\Phi\rangle$  resulting from the von Neumann measurement of a projection operator  $\hat{A}$  performed upon an ensemble of quantum systems in the preselected state  $|\psi\rangle$  depends upon  $\hat{A}|\psi\rangle$ . Here it is shown that the pointer state  $|\Psi\rangle$  obtained from such a measurement performed upon an ensemble that is also postselected depends upon the weak value of  $\hat{A}$  - *regardless of the measurement interaction strength*. It is also found that, while the spatial distribution of  $|\Psi\rangle$  exhibits interference, *the idempotency of  $\hat{A}$  prohibits interference* in that of  $|\Phi\rangle$ . This is explained in terms of *welcher Weg* information.

## I. INTRODUCTION

The weak value  $A_w$  of a quantum mechanical observable  $A$  is the statistical result of a standard measurement procedure performed upon a preselected and postselected (PPS) ensemble of quantum systems when the interaction between the measurement apparatus and each system is sufficiently weak - i.e., when the measurement is a weak measurement [1–3]. Unlike a standard strong measurement of  $A$  which significantly disturbs the measured system and yields the mean value of the associated operator  $\hat{A}$  as the observable's measured value, a weak measurement of  $A$  performed upon a PPS system does not appreciably disturb the quantum system and yields  $A_w$  as the observable's measured value. The peculiar nature of the virtually undisturbed quantum reality that exists between the boundaries defined by the PPS states is revealed by the eccentric characteristics of  $A_w$ , namely that  $A_w$  can be complex valued and that its real part can lie far outside the eigenvalue spectral limits of  $\hat{A}$ . While the interpretation of weak values remains somewhat controversial, experiments have verified several of the unusual properties predicted by weak value theory [4–10].

The impetuses for writing this note are discussions appearing in the recent literature concerning: (a) the ubiquitous and universal nature of weak values, [11, 12]; (b) the production of weak values without weak measurements [13]; and (c) the exact all order theory for weak measurements of operators  $\hat{A}$  which satisfy  $\hat{A}^2 = \hat{1}$  [14]. Here, in deference to items (a) - (c), the exact pointer theories for arbitrarily strong von Neumann measurements of both preselected (PS) and PPS systems are developed for operators satisfying  $\hat{A}^2 = \hat{A}$  (i.e. for projectors). These theories show that - unlike the pointer states for PS measurements which depend upon the action  $\hat{A}|\psi\rangle$  of  $\hat{A}$  upon the PS state  $|\psi\rangle$  - those for PPS measurements depend upon  $A_w$ , *regardless of the measurement interaction strength*; and that interference occurs in the spatial distribution of a PPS pointer state but is prevented from occurring in the spatial distribution of a PS pointer state by the idempotency of  $\hat{A}$ .

## II. EXACT POINTER THEORIES FOR VON NEUMANN PROJECTOR MEASUREMENTS

Projection operators are an important part of the general mathematical formalism of quantum mechanics. There has been a recent increased interest in these operators because

the measurement and interpretation of their weak values have played a central role in the theoretical and experimental resolution of foundational issues associated with the quantum box problem and Hardy's paradox, e.g [6, 15], as well as in the experimental observations of dynamical non-locality induced effects [16].

These operators are also interesting because their idempotent property provides for an exact description of the pointer state resulting from their measurement. Specifically, when an impulsive von Neumann measurement is performed upon a quantum system to determine the value of a time independent projection operator  $\hat{A}$ , the associated measurement operator can be written exactly as

$$e^{-\frac{i}{\hbar}\gamma\hat{A}\hat{p}} = \hat{1} - \hat{A} + \hat{A}\hat{S}, \quad (1)$$

where use has been made of the fact that  $\hat{A}^n = \hat{A}$ ,  $n \geq 1$ . Here  $\hat{p}$  is the pointer momentum operator conjugate to the position operator  $\hat{q}$ ,  $\gamma$  is the measurement interaction strength, and  $\hat{S} \equiv e^{-\frac{i}{\hbar}\gamma\hat{p}}$  is the pointer position translation operator defined by its action  $\langle q|\hat{S}|\phi\rangle \equiv \phi(q - \gamma)$  upon the initial pre-measurement pointer state  $|\phi\rangle$  (it is hereafter assumed that  $[\hat{A}, \hat{S}] = 0$ ).

### A. PS Systems

As a consequence of eq.(1), the exact normalized pointer state resulting from a measurement at time  $t$  of a quantum system prepared in the normalized PS state  $|\psi\rangle$  is

$$|\Phi\rangle = e^{-\frac{i}{\hbar}\gamma\hat{A}\hat{p}} |\psi\rangle |\phi\rangle = (\hat{1} - \hat{A} + \hat{A}\hat{S}) |\psi\rangle |\phi\rangle \quad (2)$$

(the normalization of  $|\Phi\rangle$  follows directly from the fact that  $(\hat{1} - \hat{A} + \hat{A}\hat{S})^{-1} = (\hat{1} - \hat{A} + \hat{A}\hat{S}^\dagger)^\dagger = (\hat{1} - \hat{A} + \hat{A}\hat{S}^\dagger)$ ). The associated exact spatial distribution profile  $|\langle q|\Phi\rangle|^2$  of the pointer is given by

$$|\langle q|\Phi\rangle|^2 = \left(1 - \langle\psi|\hat{A}|\psi\rangle\right) |\langle q|\Phi\rangle|^2 + \langle\psi|\hat{A}|\psi\rangle \left|\langle q|\hat{S}|\phi\rangle\right|^2, \quad (3)$$

and is simply the weighted sum of the distribution profiles for the pre-measurement state  $|\phi\rangle$  and  $\hat{S}|\phi\rangle$  - the pre-measurement state translated by  $\gamma$ . Observe that *the idempotency of  $\hat{A}$  precludes the existence of an interference cross term* proportional to  $\text{Re} \langle q|\phi\rangle^* \langle q|\hat{S}|\phi\rangle$  in eq.(3) because the cross terms contain  $\langle\psi|\hat{A}(\hat{1} - \hat{A})|\psi\rangle = \langle\psi|(\hat{A} - \hat{A}^2)|\psi\rangle = \langle\psi|(\hat{A} - \hat{A})|\psi\rangle = 0$  and  $\langle\psi|(\hat{1} - \hat{A})\hat{A}|\psi\rangle = 0$  as factors.

## B. PPS Systems

Now suppose that a measurement of projector  $\hat{A}$  is performed at time  $t$  upon a PPS system. Then the exact normalized pointer state immediately following the postselection measurement is given by

$$|\Psi\rangle = \frac{e^{i\chi}}{N} \left(1 - A_w + A_w \hat{S}\right) |\phi\rangle, \quad (4)$$

where  $|\psi_i\rangle$  and  $|\psi_f\rangle$ ,  $\langle\psi_f|\psi_i\rangle \neq 0$ , are the normalized pre- and postselected states at  $t$ , respectively;  $A_w$  is the weak value of  $A$  at  $t$  defined by

$$A_w = \frac{\langle\psi_f|\hat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle};$$

$\chi$  is the Pancharatnam phase defined by [17]

$$e^{i\chi} = \frac{\langle\psi_f|\psi_i\rangle}{|\langle\psi_f|\psi_i\rangle|};$$

and

$$N = \sqrt{|1 - A_w|^2 + |A_w|^2 + 2 \operatorname{Re} [A_w (1 - A_w^*) \langle\phi|\hat{S}|\phi\rangle]}.$$

The exact expression for the pointer's spatial probability distribution profile is

$$|\langle q|\Phi\rangle|^2 = \left(\frac{1}{N^2}\right) \left\{ |1 - A_w|^2 |\phi(q)|^2 + |A_w|^2 |\phi(q - \gamma)|^2 + 2 \operatorname{Re} [A_w (1 - A_w^*) \phi(q)^* \phi(q - \gamma)] \right\}. \quad (5)$$

The effect of postselection upon pointer states can be seen by comparing eqs.(2) and (4). *Eventhough the measurements are generally not weak measurements*, it is interesting that - unlike projector measurement pointer states for PS systems which depend upon  $\hat{A}|\psi\rangle$  - *projector measurement pointer states for PPS systems depend explicitly upon the projector's weak value  $A_w$ .*

Comparison of eqs.(3) and (5) also shows that - in addition to being a weighted sum of distribution profiles for  $|\phi\rangle$  and  $\hat{S}|\phi\rangle$  - *the pointer state distribution profile for PPS systems contains interference cross terms that are induced by state postselection.* Interference occurs here because postselection nullifies the idempotency of  $\hat{A}$  by replacing  $\hat{A}|\psi\rangle$  with  $A_w$  - thereby allowing the cross terms to occur. More specifically - unlike a PS measurement where cross terms contain the vanishing  $\langle\psi|\hat{A}(\hat{1} - \hat{A})|\psi\rangle$  and  $\langle\psi|(\hat{1} - \hat{A})\hat{A}|\psi\rangle$  factors - the cross terms for a PPS measurement contain  $A_w(1 - A_w^*)$  and its complex conjugate as non-vanishing factors.

### III. CLOSING REMARKS

The fact that PPS pointer states produced by von Neumann projector measurements of arbitrary interaction strength depend upon the weak value of the projector is - perhaps - not surprising in light of the recent discussions in [11, 12] concerning von Neumann measurements and the associated ubiquitous and universal nature of weak values. It is also interesting to note from the comparison of eqs.(2) and (4) that PPS pointer states contain a Pancharatnam phase factor. This is an expected natural consequence of state postselection [17–19].

Eqs.(2) and (4) can also be used to determine additional differences between the pointers for PS and PPS systems. For example, it is easy to show that although pointer momentum is not in general a constant of the motion for von Neumann projector measurements of PPS systems, it is a constant of the motion for PS systems (in fact this is also true for PS systems when  $\hat{A}$  is not a projector since  $[\hat{p}, e^{-\frac{i}{\hbar}\gamma\hat{A}\hat{p}}] = 0 \Rightarrow \langle\phi|\langle\psi| e^{\frac{i}{\hbar}\gamma\hat{A}\hat{p}}\hat{p}e^{-\frac{i}{\hbar}\gamma\hat{A}\hat{p}}|\psi\rangle|\phi\rangle = \langle\phi|\hat{p}|\phi\rangle$ ).

Perhaps the most interesting difference revealed by this analysis is related to interference and can be explained in terms of *welcher Weg* information. In particular, PS pointer states for projector measurements contain *welcher Weg* information in the sense that the states  $|\phi\rangle$  and  $\hat{S}|\phi\rangle$  that are superposed to form a PS pointer state are "tagged" by the vector quantities  $(\hat{1} - \hat{A})|\psi\rangle$  and  $\hat{A}|\psi\rangle$ , respectively. As shown above, the idempotency of  $\hat{A}$  naturally precludes the occurrence of PS pointer state interference. However, postselection effectively "erases" this *welcher Weg* information by replacing the vector tags with complex valued weak values of  $\hat{A}$  - thereby enabling PPS pointer states to exhibit interference.

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